

# FREDMAN'S RECIPROCITY, INVARIANTS OF ABELIAN GROUPS, AND THE PERMANENT OF THE CAYLEY TABLE

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**ABSTRACT.** Let  $\mathcal{R}$  be the regular representation of a finite abelian group  $G$  and let  $\mathcal{C}_n$  denote the cyclic group of order  $n$ . For  $G = \mathcal{C}_n$ , we compute the Poincaré series of all  $\mathcal{C}_n$ -isotypic components in  $\mathcal{S}^m \mathcal{R} \otimes \wedge^i \mathcal{R}$  (the symmetric tensor exterior algebra of  $\mathcal{R}$ ). From this we derive a general reciprocity and some number-theoretic identities. This generalises results of Fredman and Elashvili–Jibladze. Then we consider the Cayley table,  $\mathcal{M}_G$ , of  $G$  and some generalisations of it. In particular, we prove that the number of formally different terms in the permanent of  $\mathcal{M}_G$  equals  $(S^n \mathcal{R})^G$ , where  $n$  is the order of  $G$ .

## 1. INTRODUCTION

In the beginning of the 1970s, M. Fredman [7] considered the problem of computing the number of vectors  $(\lambda_0, \lambda_1, \dots, \lambda_{n-1})$  with non-negative integer components that satisfy

$$(1.1) \quad \lambda_0 + \dots + \lambda_{n-1} = m \quad \text{and} \quad \sum_{j=0}^{n-1} j\lambda_j \equiv i \pmod{n}.$$

He denoted this number by  $S(n, m, i)$ . Using generating functions, Fredman obtained an explicit formula for  $S(n, m, i)$ , which immediately showed that  $S(n, m, i) = S(m, n, i)$ . The latter is said to be *Fredman's reciprocity*. Using a necklace interpretation, he also constructed a bijection between the vectors enumerated by  $S(n, m, i)$  and those enumerated by  $S(m, n, i)$ . However, these results did not attract attention and remained unnoticed.

Later, Elashvili and Jibladze [4, 5] (partly with Patariaia [6]) rediscovered these results using Invariant Theory. Let  $\mathcal{C}_n \simeq \mathbb{Z}/n\mathbb{Z}$  be the cyclic group of order  $n$  and  $\mathcal{R}$  the space of its regular representation over  $\mathbb{C}$ . Choose a basis  $(v_0, v_1, \dots, v_{n-1})$  for  $\mathcal{R}$  consisting of  $\mathcal{C}_n$ -eigenvectors. More precisely, if  $\gamma \in \mathcal{C}_n$  is a generator and  $\zeta = \sqrt[n]{1}$  a fixed primitive root of unity, then  $\gamma \cdot v_i = \zeta^i v_i$ . Write  $\chi_i$  for the linear character  $\mathcal{C}_n \rightarrow \mathbb{C}^\times$  that takes  $\gamma$  to  $\zeta^i$ . The monomial  $v_0^{\lambda_0} \dots v_{n-1}^{\lambda_{n-1}}$  has degree  $m$  and weight  $\chi_i$  if and only if  $(\lambda_0, \lambda_1, \dots, \lambda_{n-1})$  satisfies (1.1). Thus,  $S(n, m, i)$  is the dimension of the space of  $\mathcal{C}_n$ -semi-invariants of weight  $\chi_i$  in the  $m$ th symmetric power  $\mathcal{S}^m \mathcal{R}$ . This space can also be understood as the  $\mathcal{C}_n$ -isotypic component of type  $\chi_i$  in  $\mathcal{S}^m \mathcal{R}$ , denoted by  $(\mathcal{S}^m \mathcal{R})_{\mathcal{C}_n, \chi_i}$ . To stress the connection with cyclic

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groups, we will write  $a_i(\mathcal{C}_n, m)$  in place of  $S(n, m, i)$ . The celebrated Molien formula provides a closed expression for the generating function (Poincaré series)

$$\mathcal{F}((\mathcal{S}\mathcal{R})_{\mathcal{C}_n, \chi_i}; t) = \sum_{m=0}^{\infty} a_i(\mathcal{C}_n, m) t^m,$$

where  $(\mathcal{S}\mathcal{R})_{\mathcal{C}_n, \chi_i} = \bigoplus_{m \geq 0} (\mathcal{S}^m \mathcal{R})_{\mathcal{C}_n, \chi_i}$  is the  $(\mathcal{C}_n, \chi_i)$ -isotypic component in  $\mathcal{S}\mathcal{R}$ . Then extracting the coefficient of  $t^m$  yields a formula for  $a_i(\mathcal{C}_n, m)$ , see (2.2). It is worth stressing that Molien's formula is a very efficient tool that provides a uniform approach to various combinatorial problems and paves the way for further generalisations, see e.g. [12].

In this note, we elaborate on two topics. *First*, generalising results of Fredman and Elashvili-Jibladze, we compute the Poincaré series for each  $\mathcal{C}_n$ -isotypic component in the bi-graded algebra  $\mathcal{S}\mathcal{R} \otimes \wedge^* \mathcal{R}$  and then  $\dim(\mathcal{S}^p \mathcal{R} \otimes \wedge^m \mathcal{R})_{\mathcal{C}_n, \chi_i}$  for all  $p, m, i$  (Theorem 3.2). From this we derive a more general reciprocity, see (3.5). As a by-product of these computations, we obtain some interesting identities, e.g.,

$$\exp\left(\frac{z}{1-z^2}\right) = \prod_{d=1}^{\infty} (1+z^d)^{\varphi(d)/d},$$

where  $\varphi$  is Euler's totient function. In Section 4, several identities related to isotypic components in  $\wedge^* \mathcal{R}$  are given; some of them are valid for an arbitrary finite abelian group  $G$ , see Theorem 4.4. *Second*, in Section 5, we study some properties of the Cayley table,  $\mathcal{M}_G$ , of  $G$ . If  $G = \{x_0, x_1, \dots, x_{n-1}\}$ , then  $\mathcal{M}_G$  can be regarded as  $n$  by  $n$  matrix with entries in  $\mathbb{C}[x_0, \dots, x_{n-1}] \simeq \mathcal{S}\mathcal{R}$ . For  $G = \mathcal{C}_n$ ,  $\mathcal{M}_G$  is nothing but a generic *circulant matrix*. The permanent of  $\mathcal{M}_G$ ,  $\text{per}(\mathcal{M}_G)$ , is a sum of monomials in  $x_i$ 's of degree  $n$ . Using [8], we prove that the number of different monomials occurring in this sum equals  $\dim(\mathcal{S}^n \mathcal{R})^G$ . Then we introduce the extended Cayley table,  $\tilde{\mathcal{M}}_G$  (which is a matrix of order  $n+1$ ), and characterise the monomials occurring in  $\text{per}(\tilde{\mathcal{M}}_G)$  (Theorem 5.8). This characterisation implies that the number of different monomials in  $\text{per}(\tilde{\mathcal{M}}_G)$  equals  $\dim(\mathcal{S}^{n+1} \mathcal{R})^G$ . Both  $\text{per}(\mathcal{M}_G)$  and  $\det(\mathcal{M}_G)$  belong to  $\mathcal{S}^n \mathcal{R}$ , and we prove that  $\text{per}(\mathcal{M}_G)$  is  $G$ -invariant, whereas  $\det(\mathcal{M}_G)$  is a semi-invariant whose weight is the sum of all elements of the dual group  $\hat{G}$ . The latter means that in many cases  $\det(\mathcal{M}_G)$  is invariant, too. In Section 6, we discuss some open problems related to  $(\mathcal{S}\mathcal{R})^G$  and  $\text{per}(\mathcal{M}_G)$ .

**Notation:**  $\#(M)$  is the cardinality of a finite set  $M$ ;  $(n, m)$  is the greatest common divisor of  $n, m \in \mathbb{N}$ ;  $G$  is always a finite group.

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## 2. PRELIMINARIES

**2.1. Ramanujan's sums.** Two important number-theoretic functions are *Euler's totient function*  $\varphi$  and the *Möbius function*  $\mu$ . Recall that  $\varphi(n)$  is the number of all primitive roots of unity of order  $n$ . Given  $i, n \in \mathbb{N}$ ,  $n \geq 1$ , the *Ramanujan's sum*,  $c_n(i)$ , is the sum of  $i$ -th powers of the primitive roots of unity of order  $n$ . In particular,  $c_n(0) = \varphi(n)$ . There are two useful expressions for Ramanujan's sums:

$$c_n(i) = \sum_{d|(n,i)} \mu\left(\frac{n}{d}\right) d, \quad c_n(i) = \frac{\varphi(n)}{\varphi\left(\frac{n}{(n,i)}\right)} \cdot \mu\left(\frac{n}{(n,i)}\right),$$

see [9, Theorems 271 & 272]. These formulae also show that  $c_n(1) = \mu(n)$ ,  $c_n(i) = c_n(n-i)$ , and  $c_n(i)$  is always a rational integer.

**2.2. Molien's formula for the symmetric algebra.** Let  $G$  be a finite group and  $V$  a finite-dimensional  $G$ -module. The original Molien formula computes the Poincaré series of the graded algebra of invariants  $(\mathcal{S}V)^G = \bigoplus_{m \geq 0} (\mathcal{S}^m V)^G$ . More generally, there is a similar formula for the Poincaré series of any  $G$ -isotypic component in  $\mathcal{S}V$ . Let  $\chi$  be an irreducible representation of  $G$  and  $(\mathcal{S}V)_{G,\chi}$  the isotypic component of type  $\chi$  in  $\mathcal{S}V$ . By definition, the Poincaré series of  $(\mathcal{S}V)_{G,\chi}$  is the power series  $\mathcal{F}((\mathcal{S}V)_{G,\chi}; t) := \sum_{m \geq 0} \dim((\mathcal{S}^m V)_{G,\chi}) t^m$ . Then

$$\mathcal{F}((\mathcal{S}V)_{G,\chi}; t) = \frac{\deg(\chi)}{\#(G)} \sum_{\gamma \in G} \frac{\text{tr}(\chi(\gamma^{-1}))}{\det_V(\mathbb{I} - t\gamma)},$$

see e.g. [12, Thm. 2.1]. Here  $\mathbb{I}$  is the identity matrix in  $GL(V)$ . (The algebra of invariants corresponds to the trivial one-dimensional representation, i.e., if  $\deg(\chi) = 1$  and  $\chi(\gamma) = 1$  for all  $\gamma \in G$ .)

Let  $\mathcal{R}$  be the space of the regular representation of  $G$ . For the  $G$ -module  $\mathcal{R}$ , Molien's formula can be presented in a somewhat simpler form.

**Proposition 2.1** ([2, V.1.8]). *Let  $\varphi_G(d)$  be the number of elements of order  $d$  in  $G$ . Then*

$$\mathcal{F}((\mathcal{S}\mathcal{R})^G; t) = \frac{1}{\#(G)} \sum_{d \geq 1} \frac{\varphi_G(d)}{(1 - t^d)^{(\#G)/d}}.$$

This can easily be extended to an arbitrary  $\chi$ . If  $\text{ord}(\gamma)$  is the order of  $\gamma \in G$ , then

$$(2.1) \quad \mathcal{F}((\mathcal{S}\mathcal{R})_{G,\chi}; t) = \frac{\deg(\chi)}{\#(G)} \sum_{d|\#G} \frac{\sum_{\gamma: \text{ord}(\gamma)=d} \text{tr}(\chi(\gamma^{-1}))}{(1 - t^d)^{(\#G)/d}}.$$

In fact, we prove below a more general formula (Lemma 3.1).

**2.3. Formulae of Fredman and Elashvili-Jibladze.** Recall that  $a_i(\mathcal{C}_n, m) = \dim \mathcal{S}^m(\mathcal{R})_{\mathcal{C}_n, \chi_i}$  or, equivalently, it is the number of vectors satisfying (1.1). In particular,  $a_0(\mathcal{C}_n, m) = \dim \mathcal{S}^m(\mathcal{R})^{\mathcal{C}_n}$ . If the elements of  $\mathcal{C}_n$  are regarded as the roots of unity of order  $n$ , then  $\chi_i$  is the character  $\xi \mapsto \xi^i$ ,  $\xi \in \mathcal{C}_n$ . Here  $\varphi_{\mathcal{C}_n}(d)$  is almost Euler's totient function. That is,  $\varphi_{\mathcal{C}_n}(d) = \varphi(d)$ , if  $d|n$ ; and  $\varphi_{\mathcal{C}_n}(d) = 0$  otherwise. Using (2.1) with  $G = \mathcal{C}_n$  and  $\chi = \chi_i$ , we see that  $\deg(\chi_i) = 1$  and  $\sum_{\gamma: \text{ord}(\gamma)=d} \chi_i(\gamma^{-1}) = c_d(d-i)$ . Then extracting the coefficient of  $t^m$  yields a nice-looking formula (Fredman [7], Elashvili-Jibladze [5])

$$(2.2) \quad a_i(\mathcal{C}_n, m) = \frac{1}{n+m} \sum_{d|(n,m)} c_d(i) \binom{n/d + m/d}{n/d}.$$

*Remark 2.2.* Both Fredman's approach, see (1.1), and cyclic group interpretation presuppose that  $a_i(\mathcal{C}_n, m)$  is defined for  $n \geq 1$  and  $m \geq 0$ . But (2.2) shows that  $a_i(\mathcal{C}_n, m)$  is naturally defined for  $(n, m) \in \mathbb{N}^2$ ,  $(n, m) \neq (0, 0)$ .

It follows from (2.2) that  $a_i(\mathcal{C}_n, m) = a_i(\mathcal{C}_m, n)$ . In [4, 5, 6], this equality is named the "Hermite reciprocity". As it has no relation to Hermite and was first proved by Fredman, the term *Fredman's reciprocity* seems to be more appropriate.

From (2.2), one can derive the equality

$$(2.3) \quad \sum_{(n,m) \in \mathbb{N}^2, (n,m) \neq (0,0)} a_i(\mathcal{C}_n, m) x^n y^m = - \sum_{d=1}^{\infty} \frac{c_d(i)}{d} \log(1 - x^d - y^d).$$

(Cf. [4, Remark 2], [6, Sect. 4].)

### 3. SYMMETRIC TENSOR EXTERIOR ALGEBRA AND POINCARÉ SERIES

As above, let  $V$  be a  $G$ -module. We consider the Poincaré series of the  $G$ -isotypic components in  $\mathcal{S}V \otimes \wedge V$ . Let  $(\mathcal{S}V \otimes \wedge V)_{G, \chi}$  denote the isotypic component corresponding to an irreducible representation  $\chi$ . It is a bi-graded vector space and its Poincaré series is the formal power series

$$\mathcal{F}((\mathcal{S}V \otimes \wedge V)_{G, \chi}; s, t) = \sum_{p, m \geq 0} \dim(\mathcal{S}^p V \otimes \wedge^m V)_{G, \chi} s^p t^m.$$

(Clearly, it is a polynomial with respect to  $t$ .) It is known that

$$\mathcal{F}((\mathcal{S}V \otimes \wedge V)^G; s, t) = \frac{1}{\#G} \sum_{\gamma \in G} \frac{\det_V(\mathbb{I} + t\gamma)}{\det_V(\mathbb{I} - s\gamma)},$$

see [1, Theorem 1.33]. A similar argument provides the formula for an arbitrary  $G$ -isotypic component:

$$(3.1) \quad \mathcal{F}((\mathcal{S}V \otimes \wedge V)_{G, \chi}; s, t) = \frac{\deg(\chi)}{\#G} \sum_{\gamma \in G} \text{tr}(\chi(\gamma^{-1})) \frac{\det_V(\mathbb{I} + t\gamma)}{\det_V(\mathbb{I} - s\gamma)}.$$

For, in place of the Reynolds operator  $\frac{1}{\#(G)} \sum_{\gamma \in G} \gamma$  (the projection to the subspace of  $G$ -invariants), one should merely exploit the operator  $\frac{\deg(\chi)}{\#(G)} \sum_{\gamma \in G} \text{tr}(\chi(\gamma^{-1}))\gamma$  (the projection to the isotypic component of type  $\chi$ ).

**Lemma 3.1.** *For the regular representation  $\mathcal{R}$  of  $G$ , the right-hand side of (3.1) can be written as*

$$\frac{\deg(\chi)}{\#G} \sum_{d \geq 1} \left( \sum_{\gamma: \text{ord}(\gamma)=d} \text{tr}(\chi(\gamma^{-1})) \cdot \left( \frac{1 - (-t)^d}{1 - s^d} \right)^{(\#G)/d} \right).$$

*Proof.* If  $\gamma \in G$  is of order  $d$ , then  $\langle \gamma \rangle \simeq \mathcal{C}_d$  and each coset of  $\langle \gamma \rangle$  in  $G$  is a cycle of length  $d$  with respect to the multiplication by  $\gamma$ . Hence, in a suitable basis of  $\mathcal{R}$ , the matrix of  $\gamma$  in  $GL(\mathcal{R})$  consists of  $(\#G)/d$  diagonal

blocks  $\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$  of size  $d$ . Since  $\det \begin{bmatrix} 1 & -s & 0 & \dots & 0 \\ 0 & 1 & -s & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & -s \\ -s & 0 & 0 & \dots & 1 \end{bmatrix} = 1 - s^d$ , we obtain

$$\frac{\det_{\mathcal{R}}(\mathbb{I} + t\gamma)}{\det_{\mathcal{R}}(\mathbb{I} - s\gamma)} = \left( \frac{1 - (-t)^d}{1 - s^d} \right)^{(\#G)/d}, \text{ which proves the lemma.} \quad \square$$

Now, we apply this lemma to the regular representation of  $\mathcal{C}_n$ . Recall that the number  $\binom{a+b+c}{a, b, c}$  is defined to be  $\frac{(a+b+c)!}{a! b! c!}$ .

**Theorem 3.2.** *The Poincaré series of the  $(\mathcal{C}_n, \chi_i)$ -isotypic component equals*

$$\begin{aligned} (3.2) \quad \mathcal{F}((\mathcal{S}^p \mathcal{R} \otimes \wedge^m \mathcal{R})_{\mathcal{C}_n, \chi_i}; s, t) &= \frac{1}{n} \sum_{d|n} c_d(i) \frac{(1 - (-t)^d)^{n/d}}{(1 - s^d)^{n/d}} \\ &= \frac{1}{n} \sum_{d|n} c_d(i) \left( \sum_{a=0}^{n/d} (-1)^{(d+1)a} \binom{n/d}{a} t^{ad} \right) \left( \sum_{b \geq 0} \binom{(n/d) + b - 1}{(n/d) - 1} s^{bd} \right). \end{aligned}$$

Consequently,

$$(3.3) \quad \dim(\mathcal{S}^p \mathcal{R} \otimes \wedge^m \mathcal{R})_{\mathcal{C}_n, \chi_i} = \frac{(-1)^m}{p+n} \sum_{d|n, p, m} (-1)^{m/d} c_d(i) \binom{(n+p)/d}{m/d, p/d, (n-m)/d}.$$

*Proof.* This is a straightforward consequence of Lemma 3.1. If  $G = \mathcal{C}_n$ , then  $\deg(\chi_i) = 1$  and  $\sum_{\gamma: \text{ord}(\gamma)=d} \chi_i(\gamma^{-1}) = c_d(n-i) = c_d(i)$ , which proves (3.2).

We leave it to the reader to extract the coefficient of  $t^m s^p$  in (3.2) and obtain (3.3).  $\square$

Letting  $n = q + m$  yields a more symmetric form of (3.3):

$$(3.4) \quad \dim(\mathcal{S}^p \mathcal{R} \otimes \wedge^m \mathcal{R})_{\mathcal{C}_{q+m}, \chi_i} = \frac{(-1)^m}{p+q+m} \sum_{d|p,q,m} (-1)^{m/d} c_d(i) \binom{(m+p+q)/d}{m/d, p/d, q/d}.$$

As the right-hand side is symmetric with respect to  $p$  and  $q$ , we get an equality for dimensions of isotypic components related to the regular representations of two cyclic groups,  $(\mathcal{C}_{q+m}, \mathcal{R})$  and  $(\mathcal{C}_{p+m}, \tilde{\mathcal{R}})$ :

$$(3.5) \quad \dim(\mathcal{S}^p \mathcal{R} \otimes \wedge^m \mathcal{R})_{\mathcal{C}_{q+m}, \chi_i} = \dim(\mathcal{S}^q \tilde{\mathcal{R}} \otimes \wedge^m \tilde{\mathcal{R}})_{\mathcal{C}_{p+m}, \chi_i}.$$

For  $m = 0$ , this simplifies to Fredman's reciprocity [7, (4)]. It would be interesting to have a combinatorial interpretation of this symmetry in the spirit of Fredman's approach.

*Remark 3.3.* (1) Letting  $t = 0$  in (3.2) or  $m = 0$  in (3.3), we get known formulae for the isotypic components in the symmetric algebra of  $\mathcal{R}$ , see [4, 5]. Letting  $s = 0$  in (3.2) or  $n = 0$  in (3.3), we get interesting formulae for the isotypic components in the exterior algebra of  $\mathcal{R}$ , see the next section.

(2) If  $d$  is always odd (e.g. at least one of  $m, p, q$  is odd), then  $(-1)^{m+\frac{m}{d}} = 1$  and the right-hand side of (3.4) becomes totally symmetric with respect to  $p, q, m$ .

The following is a generalisation of (2.3):

**Proposition 3.4.**

$$\sum_{(p,q,m) \in \mathbb{N}^3, p+q+m \geq 1} \dim(\mathcal{S}^p \mathcal{R} \otimes \wedge^m \mathcal{R})_{\mathcal{C}_{q+m}, \chi_i} \cdot x^p y^q z^m = - \sum_{d=1}^{\infty} \frac{c_d(i)}{d} \log(1 - x^d - y^d + (-z)^d).$$

*Proof.* By (3.4), the left-hand side equals

$$\sum_{p+q+m \geq 1} \frac{(-1)^m}{p+q+m} \sum_{d|p,q,m} (-1)^{m/d} c_d(i) \binom{(p+q+m)/d}{p/d, q/d, m/d} x^p y^q z^m.$$

Letting  $p/d = \alpha$ ,  $q/d = \beta$ ,  $m/d = \gamma$ , we rewrite it as

$$\begin{aligned} \sum_{d=1}^{\infty} \frac{c_d(i)}{d} \sum_{\alpha+\beta+\gamma \geq 1} \frac{(-1)^\gamma}{\alpha+\beta+\gamma} \binom{\alpha+\beta+\gamma}{\alpha, \beta, \gamma} x^{\alpha d} y^{\beta d} (-z)^{\gamma d} = \\ \sum_{d=1}^{\infty} \frac{c_d(i)}{d} \sum_{k \geq 1} \left( \sum_{\alpha+\beta+\gamma=k} \frac{1}{k} \binom{k}{\alpha, \beta, \gamma} (x^d)^\alpha (y^d)^\beta (-(-z)^d)^\gamma \right) = \\ \sum_{d=1}^{\infty} \frac{c_d(i)}{d} \sum_{k \geq 1} \frac{(x^d + y^d - (-z)^d)^k}{k} = - \sum_{d=1}^{\infty} \frac{c_d(i)}{d} \log(1 - x^d - y^d + (-z)^d). \end{aligned}$$

□

Specializing the equality of Proposition 3.4, we get some interesting identities.

A) Taking  $x = y = 0$  forces that  $p = q = 0$  in the left-hand side, which leads to the equality

$$\sum_{m \geq 1} \dim(\wedge^m \mathcal{R})_{c_m, \chi_i} z^m = - \sum_{d=1}^{\infty} \frac{c_d(i)}{d} \log(1 + (-z)^d).$$

For  $i = 0$ , we have  $c_d(0) = \varphi(d)$  and  $\dim(\wedge^m \mathcal{R})^{c_m} = \begin{cases} 1, & m \text{ odd}; \\ 0, & m \text{ even}. \end{cases}$

That is,  $\frac{z}{1-z^2} = - \sum_{d=1}^{\infty} \frac{\varphi(d)}{d} \log(1 + (-z)^d)$ . Replacing  $z$  with  $-z$  and exponentiating, we finally obtain:

$$\exp\left(\frac{z}{1-z^2}\right) = \prod_{d \geq 1} (1 + z^d)^{\varphi(d)/d}.$$

B) Likewise, for  $x = z = 0$  (or just  $x = 0$  in (2.3)), we get

$$- \sum_{d=1}^{\infty} \frac{c_d(i)}{d} \log(1 - y^d) = \begin{cases} y/(1-y), & i = 0 \\ 0, & i \neq 0. \end{cases}$$

In particular,

$$\exp\left(\frac{-y}{1-y}\right) = \prod_{d \geq 1} (1 - y^d)^{\varphi(d)/d}.$$

#### 4. ON THE EXTERIOR ALGEBRA OF THE REGULAR REPRESENTATION

In case of the exterior algebra of a  $G$ -module, the Poincaré series of an isotypic component is actually a polynomial in  $t$ , which can be evaluated for any  $t$ . Here we gather some practical formulae for the regular representations and for cyclic groups.

First, using (3.1) and Lemma 3.1 with trivial  $\chi$  and  $s = 0$ , we obtain

$$\mathcal{F}((\wedge \mathcal{R})^G; t) = \frac{1}{\#(G)} \sum_{d \geq 1} \varphi_G(d) (1 - (-t)^d)^{\#(G)/d}.$$

It follows that  $\mathcal{F}((\wedge \mathcal{R})^G; t)$  always has the factor  $1 + t$  and

$$(4.1) \quad \dim(\wedge \mathcal{R})^G = \frac{1}{\#(G)} \sum_{d \text{ odd}} \varphi_G(d) 2^{\#(G)/d}.$$

Note that here  $G$  is not necessarily abelian!

**Example 4.1.** For  $G = \mathfrak{S}_3$ , we have  $\varphi_G(1) = 1$ ,  $\varphi_G(2) = 3$ , and  $\varphi_G(3) = 2$ . Therefore,

$$\mathcal{F}((\wedge \mathcal{R})^{\mathfrak{S}_3}; t) = \frac{1}{6} \left( (1+t)^6 + 3(1-t^2)^3 + 2(1+t^3)^2 \right) = 1 + t + t^2 + 4t^3 + 4t^4 + t^5.$$



For  $G = \mathcal{C}_n$ , there are precise assertions for all  $G$ -isotypic components in  $\wedge^* \mathcal{R}$ . Using Theorem 3.2 with  $s = 0$  and  $p = 0$ , we obtain

$$(4.2) \quad \mathcal{F}((\wedge^* \mathcal{R})_{\mathcal{C}_n, \chi_i}; t) = \frac{1}{n} \sum_{d|n} c_d(i) (1 - (-t)^d)^{n/d},$$

$$(4.3) \quad \dim((\wedge^m \mathcal{R})_{\mathcal{C}_n, \chi_i}) = \frac{(-1)^m}{n} \sum_{d|n, m} (-1)^{m/d} c_d(i) \binom{n/d}{m/d} =: b_i(\mathcal{C}_n, m).$$

Again, it is convenient to replace  $n$  with  $q + m$  in (4.3). Then

$$b_i(\mathcal{C}_{q+m}, m) = \dim((\wedge^m \mathcal{R})_{\mathcal{C}_{q+m}, \chi_i}) = \frac{(-1)^m}{q+m} \sum_{d|q, m} (-1)^{m/d} c_d(i) \binom{q/d + m/d}{m/d}.$$

From this we derive the following observation:

**Proposition 4.2.** *If  $q$  or  $m$  is odd, then  $b_i(\mathcal{C}_{q+m}, m) = a_i(\mathcal{C}_q, m)$  and also  $b_i(\mathcal{C}_{q+m}, m) = b_i(\mathcal{C}_{q+m}, q)$ .*

**Example 4.3.**  $b_i(\mathcal{C}_{2n-1}, n-1) = a_i(\mathcal{C}_{n-1}, n)$ , and it is the  $(n-1)$ -th Catalan number regardless of  $i$ .

*Remark.* If  $n$  is odd, then  $\wedge^n \mathcal{R}$  is the trivial  $\mathcal{C}_n$ -module and therefore  $\wedge^m \mathcal{R} \simeq \wedge^{n-m} \mathcal{R}$  as  $\mathcal{C}_n$ -modules. This “explains” the equality  $b_i(\mathcal{C}_n, m) = b_i(\mathcal{C}_n, n-m)$  for  $n$  odd.

Substituting  $t = 1$  in (4.2) yields a nice formula for dimension of the whole isotypic component:

$$(4.4) \quad \dim((\wedge^* \mathcal{R})_{\mathcal{C}_n, \chi_i}) = \frac{1}{n} \sum_{d|n, d \text{ odd}} c_d(i) 2^{n/d}.$$

For  $i = 0$ , this becomes a special case of (4.1). There is a down-to-earth interpretation of (4.4) that does not invoke Invariant Theory. As in the introduction, choose a basis  $\{v_0, v_1, \dots, v_{n-1}\}$  for  $\mathcal{R}$  such that  $v_i$  has weight  $\chi_i$ . Then

$$v_{j_1} \wedge \dots \wedge v_{j_m} \in (\wedge^m \mathcal{R})_{\mathcal{C}_n, \chi_i} \iff j_1 + \dots + j_m \equiv i \pmod{n}.$$

Consequently,  $\dim(\wedge^* \mathcal{R})_{\mathcal{C}_n, \chi_i}$  equals the number of subsets  $J \subset \{0, 1, \dots, n-1\}$  such that  $|J| \equiv i \pmod{n}$ . (Here  $|J|$  stands for the sum of elements of  $J$ .) Hence our invariant-theoretic approach proves the following purely combinatorial fact:

$$\#\{J \subset \{0, 1, \dots, n-1\} \mid |J| \equiv i \pmod{n}\} = \frac{1}{n} \sum_{d|n, d \text{ odd}} c_d(i) 2^{n/d}.$$

For  $i = 0$ , this is nothing but the number of subsets of  $\mathcal{C}_n$  summing to the neutral element (in the additive notation). We provide a similar interpretation for any abelian group.



**Theorem 4.4.** *For an abelian group  $G$ , let  $\mathcal{N}_G$  denote the number of subsets  $S$  of  $G$  such that  $|S| := \sum_{\gamma \in S} \gamma = 0 \in G$ . Then  $\mathcal{N}_G = \dim(\wedge \mathcal{R})^G = \frac{1}{\#(G)} \sum_{d \text{ odd}} \varphi_G(d) 2^{\#(G)/d}$ .*

*Proof.* In view of (4.1), only the first equality requires a proof. Let  $(z_0, \dots, z_{n-1})$  be a basis for  $\mathcal{R}$  consisting of  $G$ -eigenvectors,  $n = \#(G)$ . Here the weight of  $z_i$  is a linear character  $\chi_i$  and  $\hat{G} = \{\chi_0, \chi_1, \dots, \chi_{n-1}\}$  is the dual group of  $G$ . One of the  $\chi_i$ 's is the neutral element of  $\hat{G}$ , denoted by  $\hat{0}$  in the additive notation. Then

$$z_{j_1} \wedge \dots \wedge z_{j_m} \in (\wedge^m \mathcal{R})^G \iff \chi_{j_1} + \dots + \chi_{j_m} = \hat{0} \in \hat{G}.$$

Thus,  $\dim(\wedge \mathcal{R})^G$  equals the number of subsets of  $\hat{G}$  summing to  $\hat{0}$ . However, the groups  $\hat{G}$  and  $G$  are (non-canonically) isomorphic, hence  $\mathcal{N}_G = \mathcal{N}_{\hat{G}}$  and we are done.  $\square$

## 5. ON THE PERMANENT OF THE CAYLEY TABLE OF AN ABELIAN GROUP

In this section,  $G$  is an abelian group,  $G = \{x_0, x_1, \dots, x_{n-1}\}$ . The *Cayley table* of  $G$ , denoted  $\mathcal{M}_G = (m_{i,j})$ , can be regarded as  $n$  by  $n$  matrix with entries in the polynomial ring  $\mathbb{C}[x_0, x_1, \dots, x_{n-1}] \simeq \mathcal{SR}$ . To distinguish the addition in  $\mathbb{C}[x_0, x_1, \dots, x_{n-1}]$  and the group operation in  $G$ , the latter is denoted by  $'\dot{+}'$ . By definition,  $m_{i,j} = x_i \dot{+} x_j$ ,  $i, j = 0, \dots, n-1$ . Hence  $\mathcal{M}_G$  is a symmetric matrix. The permanent of  $\mathcal{M}_G$ ,  $\text{per}(\mathcal{M}_G)$ , is a homogeneous polynomial of degree  $n$  in  $x_i$ 's, and it does not depend on the ordering of elements of  $G$ . Let  $p(G)$  denote the number of formally different monomials occurring in  $\text{per}(\mathcal{M}_G)$ .

*Remark 5.1.* In place of the Cayley table, one can consider the matrix  $\hat{\mathcal{M}}_G$  with entries  $\hat{m}_{i,j} = x_i \ominus x_j$  (the difference in  $G$ ). Clearly,  $\hat{\mathcal{M}}_G$  is obtained from  $\mathcal{M}_G$  by rearranging the columns only (or, the rows only), using the permutation on  $G$  that takes each element to its inverse. Therefore  $\text{per}(\hat{\mathcal{M}}_G) = \text{per}(\mathcal{M}_G)$  and  $\det(\hat{\mathcal{M}}_G) = \pm \det(\mathcal{M}_G)$ . Although  $\hat{\mathcal{M}}_G$  is not symmetric in general, an advantage is that every entry on the main diagonal is the neutral elements of  $G$ .

**Example 5.2.** For  $G = \mathcal{C}_n$  and the natural ordering of its elements (i.e.,  $x_i$  corresponds to  $i$ ), one obtains a generic *circulant matrix* (the latter means that the rows are successive cyclic permutations of the first row). More precisely,  $\mathcal{M}_{\mathcal{C}_n}$  (resp.  $\hat{\mathcal{M}}_{\mathcal{C}_n}$ ) is a circulant matrix

in Hankel (resp. Toeplitz) form. For instance,  $\mathcal{M}_{\mathcal{C}_3} = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_0 \\ x_2 & x_0 & x_1 \end{pmatrix}$ . Here  $\text{per}(\mathcal{M}_{\mathcal{C}_3}) =$

$x_0^3 + x_1^3 + x_2^3 + 3x_0x_1x_2$ . Therefore  $p(\mathcal{C}_3) = 4$ .

The function  $n \mapsto p(\mathcal{C}_n)$  was studied in [3], where it was pointed out that the main result of Hall [8] shows that  $p(\mathcal{C}_n)$  equals the number of solutions to

$$\begin{cases} \lambda_0 + \cdots + \lambda_{n-1} = n, \\ \sum_{j=0}^{n-1} j\lambda_j \equiv 0 \pmod{n}. \end{cases}$$

That is,  $p(\mathcal{C}_n) = a_0(\mathcal{C}_n, n)$  in our notation. Because results of [8] apply to arbitrary finite abelian groups, one can be interested in  $p(G)$  in this more general setting. Below, we give an invariant-theoretic answer using that result of Hall.

Let  $\mathcal{S}_n$  denote the symmetric group acting by permutations on  $\{0, 1, \dots, n-1\}$ . Accordingly,  $\mathcal{S}_n$  permutes the elements of  $G$  by the rule  $\pi(x_i) := x_{\pi(i)}$ . Recall that

$$\text{per}(m_{i,j}) = \sum_{\pi \in \mathcal{S}_n} \prod_{i=0}^{n-1} m_{i,\pi(i)}.$$

For the matrix  $\mathcal{M}_G$ ,

$$\prod_{i=0}^{n-1} m_{i,\pi(i)} = \prod_{i=0}^{n-1} (x_i + x_{\pi(i)}) = \prod_{i=0}^{n-1} x_i^{k_i(\pi)} =: \mathbf{x}(\pi)$$

is a monomial in  $x_i$ 's of degree  $n$ . Note that different permutations may result in the same monomial. The following is essentially proved by M. Hall.

**Theorem 5.3** ([8, n. 3]). *A monomial  $\mathbf{m} = \prod_{i=0}^{n-1} x_i^{k_i}$  is of the form  $\mathbf{x}(\pi)$  for some  $\pi \in \mathcal{S}_n$  (i.e., occurs in  $\text{per}(\mathcal{M}_G)$ ) if and only if  $\sum_i k_i = n$  and  $k_0 x_0 + \cdots + k_{n-1} x_{n-1} = 0 \in G$ . [Of course, here  $k_i x_i$  stands for  $x_i + \cdots + x_i$  ( $k_i$  times).]*

The necessity of the conditions is easy; a non-trivial argument is required for the sufficiency, i.e., for the existence of  $\pi$ .

**Theorem 5.4.**  $p(G) = \dim \mathcal{S}^n(\mathcal{R})^G$ .

*Proof.* Let  $(z_0, \dots, z_{n-1})$  be a basis for  $\mathcal{R}$  consisting of  $G$ -eigenvectors. Recall that the weight of  $z_i$  is  $\chi_i$  and  $\hat{G} = \{\chi_0, \dots, \chi_{n-1}\}$  is the dual group. The monomial  $z_0^{k_0} \cdots z_{n-1}^{k_{n-1}} \in \mathcal{S}\mathcal{R}$  is a semi-invariant of  $G$  of weight  $k_0 \chi_0 + \cdots + k_{n-1} \chi_{n-1} \in \hat{G}$ . It follows that

$$\dim \mathcal{S}^n(\mathcal{R})^G = \{(k_0, \dots, k_{n-1}) \mid \sum_i k_i = n \text{ \& } k_0 \chi_0 + \cdots + k_{n-1} \chi_{n-1} = \hat{0}\}.$$

Modulo the passage from  $G$  to  $\hat{G}$ , these conditions coincide with those of Theorem 5.3. Since  $G \simeq \hat{G}$ , we are done.  $\square$

Our next goal is to extend these results to a certain matrix of order  $n+1$ . We begin with two assertions on  $\text{per}(\mathcal{M}_G)$ , which are of independent interest.

**Proposition 5.5.** *There is a natural action  $*$  :  $G \times \mathcal{S}_n \rightarrow \mathcal{S}_n$  such that, for  $\gamma \in G$  and  $\pi \in \mathcal{S}_n$ ,  $\text{sign}(\gamma*\pi) = \text{sign}(\pi)$  and  $\mathbf{x}(\gamma*\pi) = \mathbf{x}(\pi)$ .*

*Proof.* Every  $\gamma \in G$  determines a permutation  $\sigma_\gamma$  on  $G$  and thereby an element of  $\mathcal{S}_n$ . Namely:

$$(x_0, \dots, x_{n-1}) \xrightarrow{\sigma_\gamma} (\gamma \dot{+} x_0, \dots, \gamma \dot{+} x_{n-1}).$$

Equivalently,  $x_{\sigma_\gamma(i)} = x_i \dot{+} \gamma$ . Define the  $G$ -action on  $\mathcal{S}_n$  by  $\gamma*\pi = \sigma_\gamma\pi\sigma_\gamma$ . Hence  $\text{sign}(\gamma*\pi) = \text{sign}(\pi)$ . Recall that  $\mathbf{x}(\pi) = \prod_{i=0}^{n-1} (x_i \dot{+} x_{\pi(i)})$ . Then

$$\mathbf{x}(\gamma*\pi) = \prod_{i=0}^{n-1} (x_i \dot{+} x_{\sigma_\gamma\pi\sigma_\gamma(i)}) = \prod_{j=0}^{n-1} (x_{\sigma_\gamma^{-1}(j)} \dot{+} x_{\sigma_\gamma\pi(j)}),$$

where  $j = \sigma_\gamma(i)$ . By definition,  $x_{\sigma_\gamma\pi(j)} = x_{\pi(j)} \dot{+} \gamma$  and  $x_j = x_{\sigma_\gamma^{-1}(j)} \dot{+} \gamma$ . Thus, the linear factors of  $\mathbf{x}(\gamma*\pi)$  remain the same.  $\square$

*Remark 5.6.* Our action ‘ $*$ ’ can be regarded as a generalisation of Lehmer’s “operator  $S$ ” for circulant matrices [11, p.45], i.e., essentially, for  $G = \mathcal{C}_n$ . Using that operator Lehmer proved that, for  $n = p$  odd prime,

$$\det(\mathcal{M}_{\mathcal{C}_p}) = x_0^p + \dots + x_{p-1}^p + pF(x_0, \dots, x_{p-1}),$$

where  $F \in \mathbb{Z}[x_0, \dots, x_{p-1}]$ . We note that Lehmer’s argument applies to  $\text{per}(\mathcal{M}_{\mathcal{C}_p})$  as well.

**Proposition 5.7.** *Suppose that  $\mathbf{m}$  is a monomial in  $\text{per}(\mathcal{M}_G)$  such that  $x_k$  occurs in  $\mathbf{m}$ . If  $x_k = x_i \dot{+} x_j$  for some  $i, j$ , then there is  $\sigma \in \mathcal{S}_n$  such that  $\sigma(i) = j$  and  $\mathbf{m} = \mathbf{x}(\sigma)$ .*

*Proof.* By the assumption on  $\mathbf{m}$ , there is a  $\pi \in \mathcal{S}_n$  such that  $\mathbf{m} = \mathbf{x}(\pi)$  and  $x_k = x_\alpha \dot{+} x_\beta$  for some  $\alpha, \beta$  with  $\pi(\alpha) = \beta$ . If  $\{\alpha, \beta\} \neq \{i, j\}$ , then we have to correct  $\pi$ . Take  $\gamma \in G$  such that  $x_i \dot{+} \gamma = x_\alpha$ . Then  $x_\beta \dot{+} \gamma = x_j$  and for  $\sigma = \gamma*\pi$  we have

$$\sigma(x_i) = \sigma_\gamma\pi\sigma_\gamma(x_i) = \sigma_\gamma\pi(x_\alpha) = \sigma_\gamma(x_\beta) = x_\beta \dot{+} \gamma = x_j.$$

Thus,  $\sigma(i) = j$  and also  $\mathbf{x}(\sigma) = \mathbf{x}(\pi)$  in view of Proposition 5.5.  $\square$

The Cayley table of  $G$  is the “addition table” of all elements of  $G$ . Define the *extended Cayley table* as an  $n+1$  by  $n+1$  matrix that is the “addition table” of  $n+1$  elements of  $G$ , with the neutral element taken twice. More precisely, we assume that  $x_0 = x_n = 0$  is the neutral element of  $G$  and consider the matrix  $\tilde{\mathcal{M}}_G = (m_{i,j})$ , where  $m_{i,j} = x_i \dot{+} x_j$ ,  $i, j = 0, 1, \dots, n$ . In this context,  $\mathcal{S}_{n+1}$  is regarded as permutation group on  $\{0, 1, \dots, n\}$ . Then  $\text{per}(\tilde{\mathcal{M}}_G) = \sum_{\tilde{\pi} \in \mathcal{S}_{n+1}} \mathbf{x}(\tilde{\pi})$  is a sum of monomials of degree  $n+1$ .

**Example.**  $\tilde{\mathcal{M}}_{\mathcal{C}_3} = \begin{pmatrix} x_0 & x_1 & x_2 & x_0 \\ x_1 & x_2 & x_0 & x_1 \\ x_2 & x_0 & x_1 & x_2 \\ x_0 & x_1 & x_2 & x_0 \end{pmatrix}$ ,  $\text{per}(\tilde{\mathcal{M}}_{\mathcal{C}_3}) = 2x_0^4 + 10x_0^2x_1x_2 + 4x_0x_1^3 + 4x_0x_2^3 + 4x_1^2x_2^2$ .

**Theorem 5.8.** *The monomial  $\mathbf{m} = \prod_{i=0}^{n-1} x_i^{k_i}$  occurs in  $\text{per}(\tilde{\mathcal{M}}_G)$  if and only if*

$$\sum_i k_i = n + 1 \text{ and } k_0 x_0 + \cdots + k_{n-1} x_{n-1} = 0 \in G.$$

*Proof.* " $\Rightarrow$ ". Suppose  $\mathbf{m} = \mathbf{x}(\tilde{\pi})$  for some  $\tilde{\pi} \in \mathcal{S}_{n+1}$ . Obviously,  $\deg \mathbf{m} = n + 1$ . Next,

$$k_0 x_0 + \cdots + k_{n-1} x_{n-1} = (x_0 + x_{\tilde{\pi}(0)}) + (x_1 + x_{\tilde{\pi}(1)}) + \cdots + (x_n + x_{\tilde{\pi}(n)}) = 0,$$

since the multiset  $\{x_0, x_1, \dots, x_{n-1}, x_n = x_0\}$  is closed with respect to taking inverses.

" $\Leftarrow$ ". Suppose  $\mathbf{m}$  satisfies the conditions of the theorem.

- $k_0 > 0$ . Take  $\mathbf{m}' = x_0^{k_0-1} x_1^{k_1} \cdots x_{n-1}^{k_{n-1}}$ . Then  $\mathbf{m}'$  satisfies the conditions of Theorem 5.3. Therefore  $\mathbf{m}'$  is a monomial of  $\text{per}(\mathcal{M}_G)$  and there is a  $\pi \in \mathcal{S}_n$  such that  $\mathbf{m}' = \mathbf{x}(\pi)$ . Embed  $\mathcal{S}_n$  into  $\mathcal{S}_{n+1}$  as the subgroup preserving the last element  $n$ . Let  $\tilde{\pi}$  denote  $\pi$  considered as element of  $\mathcal{S}_{n+1}$ . Then  $\mathbf{m} = \mathbf{x}(\tilde{\pi})$ .

- $k_0 = 0$ . Choose any binomial  $x_i x_j$  in  $\mathbf{m}$  and replace it with  $(x_i + x_j)x_0 = x_k x_0$  (i.e.,  $x_i + x_j = x_k$ ). That is,  $\mathbf{m} = \mathbf{m}'' x_i x_j$  is replaced with  $\mathbf{m}'' x_k x_0 =: \mathbf{m}' x_0$ . By the previous argument, we can find  $\pi \in \mathcal{S}_n$  such that  $\mathbf{m}' = \mathbf{x}(\pi)$  and  $\mathbf{m}' x_0 = \mathbf{x}(\tilde{\pi})$ . Since  $x_k = x_i + x_j$  occurs in  $\mathbf{x}(\pi)$ , we can apply Proposition 5.7 and assume that  $\pi(i) = j$  and hence  $\tilde{\pi}(i) = j$ . Finally, we replace  $\tilde{\pi}$  with  $\tilde{\pi}\tau$ , where the transposition  $\tau \in \mathcal{S}_{n+1}$  permutes  $i$  and  $n$ . One readily verifies that  $\mathbf{x}(\tilde{\pi}\tau) = \mathbf{m}'' x_i x_j = \mathbf{m}$ .  $\square$

**Corollary 5.9.** *The number of different monomials in  $\text{per}(\tilde{\mathcal{M}}_G)$  equals  $\dim(\mathcal{S}^{n+1}\mathcal{R})^G$ .*

The proof is almost identical to that of Theorem 5.4 and left to the reader.

It follows from Frobenius' theory of group determinants (see e.g. [10, §2]) that, for abelian groups,  $\det(\mathcal{M}_G)$  is the product of linear forms in  $x_i$ 's. In case of generic circulant matrices, this fact plays an important role in [11] and [13]. For future use, we provide a quick derivation. Recall that  $G = \{x_0, x_1, \dots, x_{n-1}\}$  and  $\hat{G} = \{\chi_0, \chi_1, \dots, \chi_{n-1}\}$ . Consider the  $n$  by  $n$  complex matrix  $\mathcal{K}_G$ , with  $(\mathcal{K}_G)_{i,j} = (\chi_j(x_i))$ , and the vectors  $v_j = \sum_{i=0}^{n-1} \chi_j(x_i) x_i \in \mathcal{R}$ ,  $j = 0, 1, \dots, n-1$ .

**Proposition 5.10.** *Under the above notation, we have:*

- (1)  $v_j$  is an eigenvector of  $G$  corresponding to the weight  $\chi_j^{-1}$ ;
- (2)  $\det(\mathcal{M}_G) \cdot \det(\mathcal{K}_G) = \det(\overline{\mathcal{K}_G}) v_0 v_1 \cdots v_{n-1}$ , where 'bar' stands for the complex conjugation;
- (3)  $\det(\overline{\mathcal{K}_G}) / \det(\mathcal{K}_G)$  equals the sign of the permutation  $\pi_0 \in \mathcal{S}_n$  that takes each  $x_i$  to its inverse. Hence  $\det(\mathcal{M}_G) = \text{sign}(\pi_0) v_0 v_1 \cdots v_{n-1}$ .

*Proof.* (1) Obvious.

- (2) It is easily seen that  $(\mathcal{M}_G \cdot \mathcal{K}_G)_{i,j} = \chi_j(x_i)^{-1} v_j = \overline{\chi_j(x_i)} v_j = (\overline{\mathcal{K}_G})_{i,j} v_j$ .

(3) Assuming that  $x_0$  is the neutral element, compare the coefficient of  $x_0^n$  in both parts of the equality in (2).  $\square$

Note that  $\tilde{\mathcal{M}}_G$  has equal columns and hence  $\det(\tilde{\mathcal{M}}_G) = 0$ .

*Remark 5.11.* 1. The set of vectors  $\{v_j\}$  is closed with respect to complex conjugation, and letting  $z_j = \overline{v_j} = \sum_i \overline{\chi_j(x_i)} x_i$  one obtains the eigenvector corresponding to  $\chi_j$ .

2. The orthogonality relations for the characters imply that  $\mathcal{K}_G(\overline{\mathcal{K}_G})^t = n\mathbb{I}_n$ ; that is,  $\frac{1}{\sqrt{n}}\mathcal{K}_G$  is unitary and  $|\det(\mathcal{K}_G)|^2 = n^n$ .

For the sake of completeness, we mention some other easy properties.

**Proposition 5.12.** *Suppose  $\gamma \in G$  and  $\pi \in \mathcal{S}_n$ .*

- (1)  $\gamma \cdot \mathbf{x}(\pi) = \mathbf{x}(\pi\sigma_\gamma^{-1})$ , where  $\cdot$  stands for the natural  $G$ -action on  $\mathcal{S}^n\mathcal{R}$ ;
- (2)  $\mathbf{x}(\pi) = \mathbf{x}(\pi^{-1})$ ;
- (3)  $\text{per}(\mathcal{M}_G) \in (\mathcal{S}^n\mathcal{R})^G$ ;
- (4) If  $\hat{G}$  has a unique element of order 2, say  $\psi$ , then  $\det(\mathcal{M}_G)$  is a semi-invariant of weight  $\psi$ .  
In all other cases,  $\det(\mathcal{M}_G) \in (\mathcal{S}^n\mathcal{R})^G$ .

*Proof.* 1)  $\gamma \cdot \mathbf{x}(\pi) = \gamma \cdot \prod_{i=0}^{n-1} (x_i \dot{+} x_{\pi(i)}) = \prod_{i=0}^{n-1} (x_i \dot{+} x_{\pi(i)} \dot{+} \gamma) = \prod_{i=0}^{n-1} (x_{\sigma_\gamma(i)} \dot{+} x_{\pi(i)}) = \mathbf{x}(\pi\sigma_\gamma^{-1})$ .

2) Obvious.

3) follows from (1).

4) Proposition 5.10 shows that  $\det(\mathcal{M}_G)$  is a semi-invariant whose weight equals the sum of all elements of  $\hat{G}$ . The sum of all elements of an abelian group is known to be the neutral element unless the group has a unique element of order 2, in which case the sum is this unique element.  $\square$

Note that  $\text{per}(\tilde{\mathcal{M}}_G)$  is an element of  $\mathcal{S}^{n+1}\mathcal{R}$ , but it does not belong to  $(\mathcal{S}^{n+1}\mathcal{R})^G$ .

## 6. SOME OPEN PROBLEMS

Associated with previous results on  $\text{per}(\mathcal{M}_G)$ , there are some interesting problems. Let  $d(G)$  denote the number of different monomials in  $\det(\mathcal{M}_G)$ . In view of possible cancellations, we have  $d(G) \leq p(G)$ . Using the factorisation of  $\det(\mathcal{M}_{\mathcal{C}_n})$  and theory of symmetric functions, Thomas [13] proved that  $d(\mathcal{C}_n) = p(\mathcal{C}_n)$  whenever  $n$  is a prime power. He also computed these values up to  $n = 12$  (e.g.  $d(\mathcal{C}_6) = 68 < 80 = p(\mathcal{C}_6)$ ) and suggested that the converse could be true.

**Problem 1.** *What are necessary/sufficient conditions on a finite abelian group  $G$  for the equality  $d(G) = p(G)$ ? Specifically, is it still true that the condition ' $\#(G)$  is a prime power' is sufficient?*

The equality  $\det(\mathcal{M}_G) = \text{sign}(\pi_0)v_0v_1 \dots v_{n-1}$  might be helpful in resolving Problem 1. The following problem is more general and vague.

**Problem 2.** *Let  $\mathfrak{m}$  be a monomial that satisfies conditions of Theorem 5.3. Is there a group-theoretic (or invariant-theoretic) interpretation of the coefficient of  $\mathfrak{m}$  in  $\text{per}(\mathcal{M}_G)$  or  $\det(\mathcal{M}_G)$ ?*

For  $G = \mathcal{C}_2 \oplus \mathcal{C}_2$ , we have  $p(G) = d(G) = 11$ . Because  $p(\mathcal{C}_4) = d(\mathcal{C}_4) = 10$ , one may speculate that  $p(G) \geq p(\mathcal{C}_n)$  and  $d(G) \geq d(\mathcal{C}_n)$  if  $\#(G) = n$ . By Theorem 5.4,  $p(G)$  is the coefficient of  $t^n$  in the Poincaré series  $\mathcal{F}((\mathcal{S}\mathcal{R})^G; t)$ , and one can consider a related

**Problem 3.** *Is it true that  $[t^m]\mathcal{F}((\mathcal{S}\mathcal{R})^G; t) \geq [t^m]\mathcal{F}((\mathcal{S}\mathcal{R})^{\mathcal{C}_n}; t)$  for any  $m \in \mathbb{N}$ ?*

Given  $G = \{x_0, \dots, x_{n-1}\}$ , a family of matrices  $\mathcal{M}_{G,l} \in \text{Mat}_l(\mathbb{C}[x_0, \dots, x_{n-1}])$ ,  $l \geq n$ , is said to be *admissible*, if  $\mathcal{M}_{G,n} = \mathcal{M}_G$ ,  $\mathcal{M}_{G,l}$  is a principal submatrix of  $\mathcal{M}_{G,l+1}$ , and the number of different monomials in  $\text{per}(\mathcal{M}_{G,l})$  equals  $\dim(\mathcal{S}^l \mathcal{R})^G$ .

**Problem 4.** *For what  $G$ , does an admissible family exist?*

So far, we only have matrices  $\mathcal{M}_{G,l}$  for  $l = n, n+1$ . It is possible to jump up to  $l = 2n$  by letting  $\mathcal{M}_{G,2n} = \begin{pmatrix} \mathcal{M}_G & \mathcal{M}_G \\ \mathcal{M}_G & \mathcal{M}_G \end{pmatrix}$ . It is the addition table for two consecutive sets of group elements, and it can be proved that  $\text{per}(\mathcal{M}_{G,2n})$  has the required property. Then, similarly to the construction of the extended Cayley table, one defines a larger matrix  $\mathcal{M}_{G,2n+1}$ . This procedure can be iterated, so one obtains a suitable collection of matrices of orders  $kn, kn+1$ ,  $k \in \mathbb{N}$ . However, it is not clear whether it is possible to define matrices  $\mathcal{M}_{G,l}$  for all other  $l$ . Maybe the reason is that, for arbitrary abelian  $G$ , there is no natural ordering of its elements. But, for a cyclic group, one does have a natural ordering, and we provide a conjectural definition of an admissible family of matrices.

For  $G = \mathcal{C}_n$ , it will be convenient to begin with the circulant matrix in the Toeplitz form, see Example 5.2. That is to say, our initial matrix is  $\hat{\mathcal{M}}_{\mathcal{C}_n} = (\hat{m}_{i,j})$ , where  $\hat{m}_{i,j} = x_{i-j}$ ,  $i, j = 0, 1, \dots, n-1$ , and the subscripts of  $x$ 's are interpreted  $(\text{mod } n)$ . For any  $l \geq n$ , we then define the entries of  $\hat{\mathcal{M}}_{\mathcal{C}_n,l}$  by the same formula, only the range of  $i, j$  is extended. In particular,  $\hat{\mathcal{M}}_{\mathcal{C}_n,l}$  is a Toeplitz matrix for any  $l$ .

**Example.**  $\hat{\mathcal{M}}_{\mathcal{C}_{3,5}} = \begin{pmatrix} x_0 & x_1 & x_2 & x_0 & x_1 \\ x_2 & x_0 & x_1 & x_2 & x_0 \\ x_1 & x_2 & x_0 & x_1 & x_2 \\ x_0 & x_1 & x_2 & x_0 & x_1 \\ x_2 & x_0 & x_1 & x_2 & x_0 \end{pmatrix}$

**Conjecture 6.1.** For  $l \geq n$ , the monomial  $x_0^{\lambda_0} x_1^{\lambda_1} \dots x_{n-1}^{\lambda_{n-1}}$  occurs in  $\text{per}(\hat{\mathcal{M}}_{C_n, l})$  if and only if

$$(*) \quad \lambda_0 + \dots + \lambda_{n-1} = l \quad \text{and} \quad \sum_{j=1}^{n-1} j\lambda_j \equiv 0 \pmod{n}.$$

In particular, the number of different monomials in  $\text{per}(\hat{\mathcal{M}}_{C_n, l})$  equals  $a_0(C_n, l)$ .

It is not hard to verify the necessity of  $(*)$  and that the conjecture is true for  $n = 2$ .

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